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An effective inclusion model for effective moduli of heterogeneous materials with ellipsoidal inhomogeneities

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Abstract

In the present study, an effective inclusion model for effective elastic moduli of heterogeneous materials is proposed to analyze the problem of an infinite matrix containing an ellipsoidal RVE. It is assumed that the strain energy changes of the infinite matrix due to embedding the RVE and its effective inclusion into the matrix are identical. A system of equations for effective moduli is then formulated using the new energy balance equation that interrelate effective moduli to a problem of an infinite matrix containing N inhomogeneities in an ellipsoidal sub-region. A generalized non-interacting solution derived based on the present formulation coincides with the estimates of the Hashin–Shtrikman type obtained by Ponte Casta ñ eda and Willis [J. Mech. Phys. Solids 43 (1995) 1919]. The effect of the shapes of RVEs on the approximate solution is also discussed in detail. As further application of the present formulation, the numerical models for the effective moduli of solids with cracks and heterogeneous materials with spherical inhomogeneities are proposed, which account for the interactions among many cracks or spherical inhomogeneities. The numerical results are then compared with the existing micromechanics models and experimental data. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Heterogeneous materials; Effective moduli; Effective inclusion model; Energy balance equation; Generalized non-interacting solution; Numerical model

1. Introduction

Many approximate models and effective medium theories for effective elastic moduli of heterogeneous materials with randomly dispersed inhomogeneities have been proposed (see reviews by Hashin, 1983; Christensen, 1990; Kachanov, 1992; Nemat-Nasser and Hori, 1993). The homogenization procedures used in these micromechanics models usually simplify the complex geometry of the original heterogeneous materials into the problem of a homogeneous media with only one inhomogeneity.

Zimmerman (1991) presented a closed-form solution for the differential scheme equations for the effective elastic moduli of materials containing spherical pores or rigid spheres. In addition, he showed that

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the experimental data obtained by Walsh et al. (1965) and Hasselman and Fulrath (1965) lie between the predictions of the Mori–Tanaka solution and the differential method and somewhat closer to those of the latter. The experimental data of porous materials obtained by Walsh et al. (1965) and Ishai and Cohen (1967) were compared with the theoretical predictions by Cleary et al. (1980) and Weng (1984). Again, these data lie between the predictions of the Mori–Tanaka solution and the differential method. Christensen (1990) tested the effective moduli theories using the measurement of the viscosity of Newtonian fluids containing rigid spheres. However, as pointed out by Zimmerman (1991), it is problematic to use the measurements of relative viscosity of a suspension of rigid spheres in a Newtonian fluid since the dispersed spheres tend to migrate away from walls, creating an inhomogeneous suspension when the viscosity of a suspension is measured.

Numerical methods have been used to predict the overall mechanical properties of heterogeneous materials (Kachanov, 1987, 1992; Rodin and Hwang, 1991; Day et al., 1992; Huang et al., 1996; Yi et al., 1995; Shen and Yi, 2000a). For two-dimensional (2-D) problem of cracked solids, Kachanov (1987) presented a simple method to analyze the problem of an infinite matrix with a finite number of cracks, and derived explicit expressions for solids with cracks that can consider the crack arrays within an elementary volume. Kachanov (1992) also gave the numerical calculations for 2-D problem of solids with randomly located cracks. His results are slightly larger than the predictions of the conventional non-interaction solution. A similar calculation for the case of anisotropic matrix was also given by Mauge and Kachanov (1994). Huang et al. (1994) pointed out that the solutions by Kachanov (1987, 1992) neglected the interaction between cracks outside and inside the square elementary volume, and proposed an approximate scheme to account for the interaction by adding another layer of cracks outside the square elementary volume. Furthermore, Huang et al. (1996) presented the boundary element method in conjunction with a unit cell model for 2-D case of solids with cracks to account for the interaction. Twenty five cracks are involved in their unit cell. Their numerical results generally lie between the predictions by the conventional non-interaction solution and the differential method. The recent work by Zhan et al. (1999) also considers the interaction based on a superposition scheme and series expansions of the complex potentials. A complicated calculation of a finite plate is involved in their method. Their numerical results are consistent with those of Huang et al. (1996). It is noted that it is difficult to extend the methods proposed by Huang et al. (1996) or Zhan et al. (1999) to three-dimensional (3-D) cases with any kind of inhomogeneities.

Rodin and Hwang (1991) extended Kachanov's method to the problem of an infinite matrix containing a finite number of spherical inhomogeneities and calculated the effective shear moduli of an impressible matrix with randomly dispersed rigid spheres. Their calculations are also consistent with the conventional non-interaction solution. Meanwhile, they explained that their calculations neglected the interactions between the inhomogeneities and the external surface of the matrix.

Essentially, the interactions pointed out by Rodin and Hwang (1991) and Huang et al. (1994) are identical. The problem involved in the calculations for effective moduli by Rodin and Hwang (1991) and Kachanov (1987, 1992) is an inconsistency between the conventional energy balance equation and the system of controlling equations for the problem of an infinite matrix containing N inhomogeneities. It is known that the conventional energy balance equation for effective moduli requires the solution of a finite block of heterogeneous material with N inhomogeneities or an infinite heterogeneous material with an infinite number of inhomogeneities. However, the problems that were calculated by Kachanov (1992) and Rodin and Hwang (1991) are an infinite intact matrix with a square block of heterogeneous material that contains a finite number of cracks or spheres.

In our recent works (Shen and Yi, 2000a,b), a new energy balance equation for effective moduli of solids with cracks has been proposed that just requires the solution of the problem of an infinite matrix containing N cracks in a spherical (3-D) or circular (2-D) sub-region. The inconsistency mentioned previously has been rigorously and analytically overcome. In other words, the interaction pointed out by Huang et al. (1994) has been rigorously and analytically accounted for. The numerical results for 2-D solids with randomly located cracks (Shen and Yi, 2000a) also lie between the predictions by the conventional non-interaction solution and differential method. Moreover, it is noted that this method is very simple and identically suitable for 2-D or 3-D problems with any kind of inhomogeneities.

Ju and Chen (1994a,b) presented the "non-interacting" solution and the second-order particle interaction model by analyzing an infinite matrix containing an ellipsoidal RVE. Hori and Nemat-Nasser (1993) presented a very general analytical solution by analyzing an infinite matrix containing a double inclusion. Nemat-Nasser and Hori (1995) also analyzed the geometry where the properties of the infinite matrix can be arbitrary. Ponte Casta ñ eda and Willis (1995) derived estimates of the Hashin–Shtrikman type. The shape of RVE was involved in these models. However, it was not explained clearly.

In the present study, an effective inclusion model for effective moduli of heterogeneous materials is proposed to analyze the problem of an infinite matrix containing an ellipsoidal RVE. Therefore, a new energy balance equation for heterogeneous materials with any kind of inhomogeneities is developed. Moreover, a generalized non-interaction solution is derived, which coincides with the estimates of the Hashin–Shtrikman type obtained by Ponte Casta ñ eda and Willis (1995). On the basis of the present formulation, the effect of shape of RVEs on effective moduli in the existing solutions can be clarified. Furthermore, numerical models for cracked solids or composites containing spherical inhomogeneities are also obtained, which account for the mutual positions of many cracks or spherical inhomogeneities. Numerical calculations for 2-D solids with parallel cracks and two-phase composites with spherical pores or rigid spheres are carried out.

2. An effective inclusion model

Fig. 1(a) and (b) shows an ellipsoidal RVE of a heterogeneous material and the effective inclusion of the RVE. The elastic moduli and geometry (shape and size) of the inclusion are the same as the effective moduli of the heterogeneous material and the RVE, respectively. As shown in Fig. 1(c) and (d), Δf_{micro} and $\Delta f_{\text{effective}}$ denote the strain energy changes of an infinite matrix, which is subjected to far-field stress σ_0 and has the same elastic moduli as the matrix of the RVE, due to embedding the heterogeneous RVE and the homogeneous effective inclusion into the infinite matrix, respectively. It is assumed

$$\Delta f_{\text{effective}} = \Delta f_{\text{micro}} \tag{1}$$

Δf_{micro} can be micromechanically determined in terms of the "microstructures" of the N inhomogeneities contained in the ellipsoidal sub-region of the infinite matrix, while $\Delta f_{\text{effective}}$ can be interrelated to the effective moduli of the RVE (Eshelby, 1957),

$$\Delta f_{\text{effective}} = -\tfrac{1}{2}V\sigma^0 : \left[\mathbf{C}_0 : (\mathbf{C} - \mathbf{C}_0)^{-1} : \mathbf{C}_0 + \mathbf{C}_0 : \mathbf{S}_V\right]^{-1} : \sigma^0 \tag{2}$$

where V is the volume of the effective inclusion or the RVE (which also denotes the corresponding sub-region occupied by the effective inclusion or the RVE); $\mathbf{S}_V$ is Eshelby's tensor associated with the material properties of the matrix and the shape of the RVE, and σ^0 accompanied by $\varepsilon^0 = \mathbf{C}_0 : \sigma^0$ is the uniform far-field stress tensor; $\mathbf{C}_0$ and $\mathbf{C}$ are the elastic stiffness tensors of the matrix and the effective medium of the heterogeneous material, respectively. If the matrix is isotropic, the explicit expression of Eshelby's tensor $\mathbf{S}_V$ is available (Eshelby, 1957, 1959; Mura, 1982).

Eqs. (2) and (1) lead to an energy balance equation

$$-\tfrac{1}{2}V\sigma^0 : \left[(\mathbf{C} - \mathbf{C}_0)^{-1} : \mathbf{C}_0 + \mathbf{S}_V\right]^{-1} : \mathbf{C}_0^{-1} : \boldsymbol{\sigma}^0 = \Delta f_{\text{micro}} \tag{3}$$

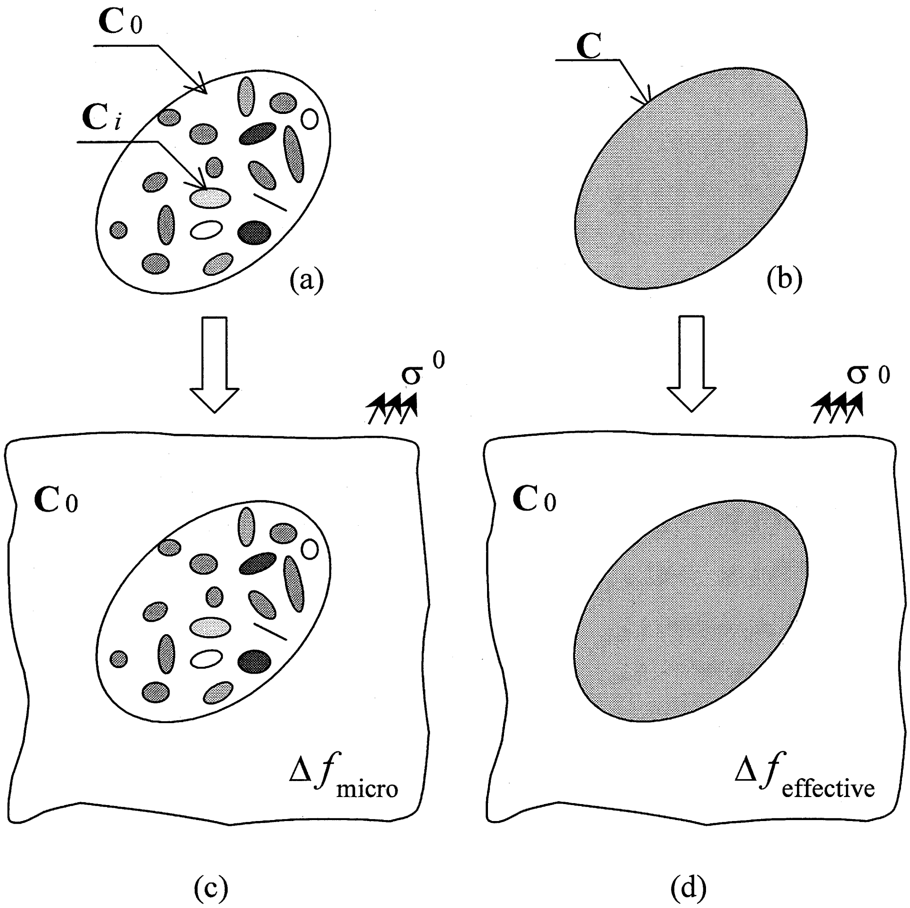

Fig. 1. Schematic diagram of the effective-inclusion model: (a) an RVE of the heterogeneous material; (b) the effective inclusion; (c) the strain energy change $\Delta f_{\mathrm{micro}}$ due to embedding the RVE into the infinite matrix; (d) the strain energy change $\Delta f_{\mathrm{effective}}$ due to embedding the effective inclusion into the infinite matrix.

The energy balance equation interrelates the effective moduli of heterogeneous materials to the strain energy change $\Delta f_{\mathrm{micro}}$ which retains the need to be micromechanically determined in terms of the "microstructural" quantities, say average strains of the N inhomogeneities contained in the sub-region.

Let $\bar{\varepsilon}_i$ and $\bar{\varepsilon}_i^*$ denote the average strain and the average eigenstrain of the ith inhomogeneity over the region V_i occupied by it. The strain energy change $\Delta f_{\mathrm{micro}}$ can then be expressed as (see Mura, 1982)

$$\Delta f_{\mathrm{micro}} = \frac{1}{2} V \sigma^0 : \sum_{i=1}^{N} \phi_i (\mathbf{I} - \mathbf{C}_0^{-1} : \mathbf{C}_i) : \bar{\varepsilon}_i \tag{4a}$$

or

$$\Delta f_{\mathrm{micro}} = \frac{1}{2} V \sigma^0 : \sum_{i=1}^{N} \phi_i \bar{\varepsilon}_i^* \tag{4b}$$

where ϕ_i with $\phi_i = V_i/V$ and $\mathbf{C}_i$ are the volume fraction and the elastic stiffness tensor of the ith inhomogeneity, respectively. In accordance with Eshelby (1957), ε_i^* is defined by

$$\mathbf{C}_i(\varepsilon^0 + \varepsilon'(\mathbf{x})) = \mathbf{C}_0 : (\varepsilon^0 + \varepsilon'(\mathbf{x}) - \varepsilon_i^*(\mathbf{x})), \quad \text{for } \mathbf{x} \text{ in } V_i \ (i = 1, 2, \ldots N) \tag{5}$$

$$\varepsilon'(\mathbf{x}) = \int_V \mathbf{s}(\mathbf{x} - \mathbf{x}') : \varepsilon^*(\mathbf{x}')\,\mathrm{d}\mathbf{x}' \tag{6}$$

where $\mathbf{s}(\mathbf{x} - \mathbf{x}')$ is a fourth-order tensor-valued function of $\mathbf{x} - \mathbf{x}'$, defined by Green's function of the infinite matrix, ε^0 and ε' are the far-field strain tensor and the perturbation strain tensor, $\varepsilon^*(\mathbf{x}) = \varepsilon_i^*(\mathbf{x})$, for $\mathbf{x}$ in V_i and otherwise $\varepsilon^*(\mathbf{x}) = 0$. By averaging Eq. (5) over V_i, the average eigenstrain $\overline{\varepsilon}_i^*$ of the ith inhomogeneity can be related to its average strain

$$\overline{\varepsilon}_i^* = (\mathbf{I} - \mathbf{C}_0^{-1} : \mathbf{C}_i) : \overline{\varepsilon}_i \tag{7}$$

Thus Eqs. (4a) and (4b) are consistent.

From Eqs. (3) and (4a) or Eq. (4b), the energy balance equations which interrelate effective moduli to the "microstructural" quantities, $\overline{\varepsilon}_i$ or $\overline{\varepsilon}_i^*$ are obtained as

$$\sigma^0 : \left[(\mathbf{C} - \mathbf{C}_0)^{-1} : \mathbf{C}_0 + \mathbf{S}_V\right]^{-1} : \mathbf{C}_0^{-1} : \sigma^0 = -\sigma^0 : \sum_{i=1}^{N} \phi_i (\mathbf{I} - \mathbf{C}_0^{-1} : \mathbf{C}_i) : \overline{\varepsilon}_i \tag{8a}$$

or

$$\sigma^0 : \left[(\mathbf{C} - \mathbf{C}_0)^{-1} : \mathbf{C}_0 + \mathbf{S}_V\right]^{-1} : \mathbf{C}_0^{-1} : \sigma^0 = -\sigma^0 : \sum_{i=1}^{N} \phi_i \overline{\varepsilon}_i^* \tag{8b}$$

Approximate analytical models for effective moduli can be obtained using approximate schemes to get the average strain $\overline{\varepsilon}_i$ of the ith inhomogeneity. The system of Eq. (8b) with Eqs. (5) and (6) can give accurate results of effective moduli which can be used to verify the validity of approximate analytical models by using an accurate method to solve the integral Eqs. (5) and (6) for $\overline{\varepsilon}_i^*$.

3. Non-interacting approximation

3.1. A generalized non-interacting solution

If the interactions among the N inhomogeneities are completely neglected, $\overline{\varepsilon}_i$ can be found by solving the problem of the infinite matrix containing the ith isolated inhomogeneity. Using Eshelby's method (Eshelby, 1957), $\overline{\varepsilon}_i$ is obtained as

$$\overline{\varepsilon}_i = -\left(\mathbf{I} - \mathbf{C}_0^{-1} : \mathbf{C}_i\right)^{-1} : \left[(\mathbf{C}_i - \mathbf{C}_0)^{-1} : \mathbf{C}_0 + \mathbf{S}_i\right]^{-1} : \mathbf{C}_0^{-1} : \sigma^0 \tag{9}$$

where $\mathbf{S}_i$ is Eshelby's tensor associated with the material properties of the matrix and the shape of the ith inhomogeneity. Substituting Eq. (9) into Eq. (8a) and considering the arbitrariness of the far-field stresses yield

$$\left[(\mathbf{C} - \mathbf{C}_0)^{-1} + \mathbf{S}_V : \mathbf{C}_0^{-1}\right]^{-1} = \sum_{i=1}^{N} \phi_i \left[(\mathbf{C}_i - \mathbf{C}_0)^{-1} + \mathbf{S}_i : \mathbf{C}_0^{-1}\right]^{-1} \tag{10}$$

Consequently, an approximate solution for effective elastic stiffness tensor $\mathbf{C}$ can be extracted as

$$\mathbf{C} = \mathbf{C}_0 + \left[\mathbf{I} - \sum_{i=1}^{N} \phi_i \mathbf{T}_i : \mathbf{S}_V : \mathbf{C}_0^{-1}\right]^{-1} : \left[\sum_{i=1}^{N} \phi_i \mathbf{T}_i\right] \tag{11}$$

with

$$\mathbf{T}_i = \left[(\mathbf{C}_i - \mathbf{C}_0)^{-1} + \mathbf{S}_i : \mathbf{C}_0^{-1}\right]^{-1} \tag{12}$$

For distinction from the conventional non-interaction solution, the present solution is called a generalized non-interaction solution. It is known that the interrelation between the $\mathbf{P}$ tensor used by Ponte Casta ñ eda and Willis (1995) and Eshelby's tensor $\mathbf{S}$ is $\mathbf{P} = \mathbf{S} : \mathbf{C}_0^{-1}$. It can be seen that the generalized non-interacting solution, i.e., Eq. (11) coincides with the estimates of the Hashin–Shtrikman type obtained by Ponte Casta ñ eda and Willis (1995).

Some special cases of the solution are as follows:

If all inhomogeneities are unidirectionally aligned and similarly shaped, Eq. (11) becomes

$$\mathbf{C} = \mathbf{C}_0 : \left[\mathbf{I} + \mathbf{B} : (\mathbf{I} - \mathbf{S}_V : \mathbf{B})^{-1}\right] \tag{13}$$

with

$$\mathbf{B} = \sum_{i=1}^{N} \left[\mathbf{s} + (\mathbf{C}_i - \mathbf{C}_0)^{-1} : \mathbf{C}_0\right]^{-1} \tag{14}$$

where $\mathbf{s}$ is the common Eshelby's tensor of the N inhomogeneities with $\mathbf{s} = \mathbf{S}_i$.

If the shape and orientation of the RVE are identical to those of the inhomogeneities, i.e. $\mathbf{S}_V = \mathbf{s}$, Eq. (13) recovers the "non-interacting" solution obtained by Ju and Chen (1994a). They also described the connection between their model and the Mori–Tanaka method. Further, if all inhomogeneities have identical moduli tensor $\mathbf{C}_1$, Eq. (13) becomes

$$\mathbf{C} = \mathbf{C}_0 : \left\{\mathbf{I} + \phi_1\left[(\mathbf{C}_1 - \mathbf{C}_0)^{-1} : \mathbf{C}_0 + (1 - \phi_1)\mathbf{s}\right]^{-1}\right\} \tag{15}$$

which is identical to the estimates based on the Mori–Tanaka method (Mori and Tanaka, 1973) for two-phase composites. The tensors in Eq. (15) are coupled. Tandon and Weng (1984) and Zhao et al. (1989) presented the individual components of the effective moduli.

If all inhomogeneities have identical moduli $\mathbf{C}_1$ in Eq. (13), it becomes

$$\mathbf{C} = \mathbf{C}_0 : \left\{\mathbf{I} + \phi_1\left[(\mathbf{C}_1 - \mathbf{C}_0)^{-1} : \mathbf{C}_0 + \mathbf{s} - \phi_1\mathbf{S}_V\right]^{-1}\right\} \tag{16}$$

where ϕ_1 is the total volume fraction of inhomogeneities (with possibly different sizes). Eq. (16) is consistent with Eq. (3.25) of Ponte Casta ñ eda and Willis (1995). Further, if $\mathbf{s}$ and $\mathbf{S}_V$ in Eq. (16) are corresponding to spheroidal inhomogeneities and the spheroidal RVE that has the same axial direction as that of inhomogeneities but may be different in the aspect ratio, the effective moduli $\mathbf{C}$ predicted by Eq. (16) is transversely isotropic. For the special cases that inhomogeneities are rigid disks or penny shaped cracks, only the two components of $\mathbf{C}$ are non-trivial, which are in-plane bulk and transverse shear moduli for the case of rigid discs, and longitudinal Young's modulus and longitudinal shear modulus for the case of cracks, respectively. Ponte Casta ñ eda and Willis (1995) explicitly gave the results for the special cases of spherical and "flat" distributions of rigid disks or cracks. Note that the spherical and "flat" distributions of rigid disks or cracks are corresponding to the spherical RVE and "flat" RVE of the present model.

3.2. Problem about the shape of RVE

Effective moduli should be independent of the shape of RVE. However, it can be seen that the generalized non-interacting solution is dependent of the shape of the RVE. In order to discuss the problem, a parameter for the interactions among the N inhomogeneities is introduced. Let $\Delta\bar{\varepsilon}_i$ denote the neglected part of the average strain of the ith inhomogeneity as assuming the non-interacting approximation to solve the $\bar{\varepsilon}_i$ (see Eq. (9)). In terms of the linear elasticity, it can be supposed that there exists an as-yet-unknown interaction tensor $\mathbf{F}_i$, which linearly interrelates $\Delta\bar{\varepsilon}_i$ to the far-field stress σ^0 as follows

$$\Delta\bar{\varepsilon}_i \equiv (\mathbf{I} - \mathbf{C}_0^{-1} : \mathbf{C}_i)^{-1} : \mathbf{F}_i : \mathbf{C}_0^{-1} : \sigma^0 \tag{17}$$

It is noted that $\mathbf{F}_i$ is associated with the shape of the sub-region V or the RVE. After considering the interactions among the N inhomogeneities, the average strain $\bar{\varepsilon}_i$ of the ith inhomogeneity should be the sum of Eqs. (9) and (17)

$$\bar{\varepsilon}_i = -(\mathbf{I} - \mathbf{C}_0^{-1} : \mathbf{C}_i)^{-1} : (\mathbf{T}_i + \mathbf{F}_i) : \mathbf{C}_0^{-1} : \sigma^0 \tag{18}$$

Thus, by substituting Eq. (18) into Eq. (8a), the expression for effective elastic stiffness tensor $\mathbf{C}$ which has accounted for the as-yet-unknown interactions among inhomogeneities is obtained as

$$\mathbf{C} = \mathbf{C}_0 + \left[\mathbf{I} - \sum_{i=1}^{N}\phi_i(\mathbf{T}_i + \mathbf{F}_i) : \mathbf{S}_V : \mathbf{C}_0^{-1}\right]^{-1} : \left[\sum_{i=1}^{N}\phi_i(\mathbf{T}_i + \mathbf{F}_i)\right] \tag{19}$$

Physically, the effective moduli $\mathbf{C}$ must be independent of the shape of the RVE. If $\mathbf{F}_i$ are exactly evaluated, the dependency of $\mathbf{F}_i$ on the shape of the RVE must cancel with that of $\mathbf{S}_V$. However, as it is extremely difficult to solve the interactions among N randomly distributed inhomogeneities exactly, approximate estimates have been used. The dependency of $\mathbf{F}_i$ on the shape of the RVE may not cancel with that of $\mathbf{S}_V$ completely. Consequently, the different shapes of RVE lead to the different approximate solutions of effective moduli $\mathbf{C}$. For example, the dependence of $\mathbf{S}_V$ on the shape of the RVE remains in the generalized non-interacting solution since $\mathbf{F}_i$ are neglected. As far as the generalized non-interacting solution is concerned, some basic requirements can be deliberately made for the choice of the shape of the RVE to have physically reasonable effective moduli $\mathbf{C}$. For isotropic problems with randomly distributed inhomogeneities, the $\mathbf{S}_V$ in the generalized non-interacting solution must be isotropic to acquire isotropic effective moduli $\mathbf{C}$, which requires that the shape of RVE should be spherical.

4. Solids with parallel tunnel cracks

For crack problems, it is convenient to rewrite the present model using displacement discontinuities across crack faces even though it can also be analyzed by taking cracks as the limited case of ellipsoidal voids. In the following, the solids with parallel tunnel cracks are analyzed.

For the plane strain problem of solids with parallel tunnel cracks (in x_1–x_2 plane and cracks are normal to x_1 axis), the non-trivial components of the effective moduli are the effective plane strain modulus E_1 in the x_1 direction and the in-plane shear modulus G_{12}.

By taking the corresponding far-field boundary stresses, i.e. uniaxial tension σ_E^0 with $\sigma_{E11}^0 = \sigma_E^0$ and other $\sigma_{E\alpha\beta}^0 = 0$, and in-plane pure shear σ_G^0 with $\sigma_{G12}^0 = \sigma_{G21}^0 = \sigma_G^0$ and other $\sigma_{E\alpha\beta}^0 = 0$, respectively, and considering circular cylinder RVE, the two independent equations for effective moduli can be obtained by

$$\frac{1}{2E_0'}\frac{E_1 - E_0'}{E_0' + \xi_E(E - E_0')} = -\frac{1}{(\sigma_E^0)^2}\frac{1}{A}\sigma_E^0 : \sum_{i=1}^{N}\frac{1}{2}\int_{l_i}(\mathbf{n}_i\mathbf{b}_i + \mathbf{b}_i\mathbf{n}_i)\,\mathrm{d}l_i \tag{20}$$

$$\frac{1}{2G_0}\frac{G_{12} - G_0}{G_0 + \eta_G(G_{12} - G_0)} = -\frac{1}{(\sigma_G^0)^2}\frac{1}{A}\sigma_G^0 : \sum_{i=1}^{N}\frac{1}{2}\int_{l_i}(\mathbf{n}_i\mathbf{b}_i + \mathbf{b}_i\mathbf{n}_i)\,\mathrm{d}l_i \tag{21}$$

where the strain energy change expressed in the right side of Eq. (8a) has been expressed in terms of the unknown displacement discontinuities $\mathbf{b}_i$ of the N cracks across the crack faces l_i; $\mathbf{n}_i$ is an unit normal to the

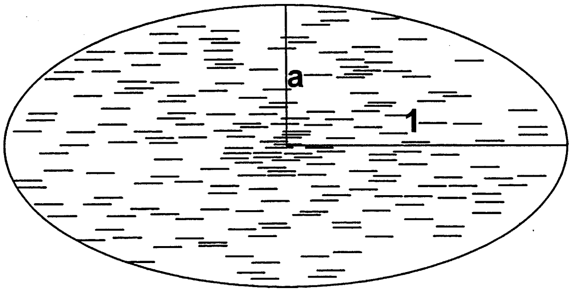

Fig. 2. An elliptic sample with 200 parallel cracks ($\rho = 0.3$ and aspect ratio $a = 0.5$).

ith crack face; and l_i denotes the region of the ith carck; E_0 and v_0 are the Young's modulus and Poisson's ratio of the intact solids; and $\xi_E = 5/8$, $\eta_G = (3 - 4v_0)/[4(1 - v_0)]$.

Furthermore, by considering elliptic RVEs with the same axial direction as the normal direction of cracks, an independent equation for the in-plane shear modulus can be found as

$$\frac{1}{2G_0}\frac{G_{12} - G_0}{G_0 + 2S_{V1212}(G_{12} - G_0)} = -\frac{1}{(\sigma_G^0)^2}\frac{1}{A}\sigma_G^0 : \sum_{i=1}^{N}\frac{1}{2}\int_{l_i}(\mathbf{n}_i\mathbf{b}_i + \mathbf{b}_i\mathbf{n}_i)\,\mathrm{d}l_i \tag{22}$$

where S_{V1212} is the component of Eshelby's tensor corresponding to the elliptic cylinder RVE, which is associated with the aspect ratio a of the elliptic cross-section of the RVE (see Fig. 2). Note that for the special case of circular RVE, $2S_{V1212} = \eta_G$.

As $\mathbf{n}_i$ is constant for each crack, $\int_{l_i}(\mathbf{n}_i\mathbf{b}_i + \mathbf{b}_i\mathbf{n}_i)\,\mathrm{d}l_i = 2l_i(\mathbf{n}_i\langle\mathbf{b}_i\rangle + \langle\mathbf{b}_i\rangle\mathbf{n}_i)$ with $\langle\mathbf{b}_i\rangle = 1/2l_i\int_{l_i}\mathbf{b}_i\,\mathrm{d}l_i$. Therefore, the average displacement discontinuity across each crack face is required in order to evaluate effective moduli of solids with tunnel cracks.

4.1. Generalized non-interacting solution

Based on the non-interacting approximation, the strain energy changes in the left sides of Eqs. (20)–(22) can be given as (Kachanov, 1994)

$$\frac{1}{A}\sigma_E^0 : \sum_{i=1}^{N}\frac{1}{2}\int_{l_i}(\mathbf{n}_i\mathbf{b}_i + \mathbf{b}_i\mathbf{n}_i)\,\mathrm{d}l_i = (\sigma_E^0)^2(\pi/E_0')\rho \tag{23}$$

$$\frac{1}{A}\sigma_G^0 : \sum_{i=1}^{N}\frac{1}{2}\int_{l_i}(\mathbf{n}_i\mathbf{b}_i + \mathbf{b}_i\mathbf{n}_i)\,\mathrm{d}l_i = (\sigma_G^0)^2(\pi/E_0')\rho \tag{24}$$

where ρ denotes the crack density with $\rho = (1/A)\sum_{i=1}^{N} l_i^2$ and A is the area of the circular or elliptic RVE. Substituting Eq. (23) into Eq. (20) and Eq. (24) into Eqs. (21) and (22) leads to

$$E_1/E_0' = 1 - \frac{2\pi\rho}{1 + 2\xi_E\pi\rho} \tag{25}$$

$$G_{12}/G_0 = 1 - \frac{\pi(1 - v_0)\rho}{1 + 2\eta_G\pi(1 - v_0)\rho} \tag{26}$$

$$G_{12}/G_0 = 1 - \frac{\pi(1 - v_0)\rho}{1 + 2S_{V1212}\pi(1 - v_0)\rho} \tag{27}$$

Eq. (27) can give different evaluations of effective in-plane shear modulus for different aspect ratios of elliptical RVEs. Note that, for 3-D case, i.e., solids containing parallel penny-shaped cracks, similar results were presented by Ponte Casta ñ eda and Willis (1995).

4.2. Numerical calculation

The unknown average displacement discontinuities $\langle \mathbf{b}_i \rangle$ are numerically calculated using Kachanov's method (Kachanov, 1987) to solve the N crack interaction problem. Kachanov (1987) presented a system of N vectorial linear algebraic equations for the average traction $\langle \mathbf{t}_i \rangle$ over the ith crack and a simple proportionality relation between $\langle \mathbf{b}_i \rangle$ and $\langle \mathbf{t}_i \rangle$,

$$\langle \mathbf{t}_i \rangle = \mathbf{t}_i^0 + \Lambda_{ik} \cdot \langle \mathbf{t}_k \rangle \tag{28}$$

$$\langle \mathbf{b}_i \rangle = \frac{\pi l_i}{E_0'} \langle \mathbf{t}_i \rangle \tag{29}$$

where the tensorial element Λ_{ik} gives the average traction vector generated along the kth crack line by the ith crack loaded by a uniform traction of arbitrary direction and unit intensity.

As a result, the present energy balance equations (20)–(22) become

$$\frac{1}{2E_0'} \frac{E_1 - E_0'}{E_0' + \xi_E(E - E_0')} = -\frac{1}{(\sigma_E^0)^2} \frac{\pi}{AE_0} \sigma_E^0 : \sum_{i=1}^{N} l_i^2 (\mathbf{n}_i \langle \mathbf{t}_i \rangle + \langle \mathbf{t}_i \rangle \mathbf{n}_i) \tag{30}$$

$$\frac{1}{2G_0} \frac{G_{12} - G_0}{G_0 + \eta_G(G_{12} - G_0)} = -\frac{1}{(\sigma_G^0)^2} \frac{\pi}{AE_0'} \sigma_G^0 : \sum_{i=1}^{N} l_i^2 (\mathbf{n}_i \langle \mathbf{t}_i \rangle + \langle \mathbf{t}_i \rangle \mathbf{n}_i) \tag{31}$$

$$\frac{1}{2G_0} \frac{G_{12} - G_0}{G_0 + 2S_{V1212}(G_{12} - G_0)} = -\frac{1}{(\sigma_G^0)^2} \frac{\pi}{AE_0'} \sigma_G^0 : \sum_{i=1}^{N} l_i^2 (\mathbf{n}_i \langle \mathbf{t}_i \rangle + \langle \mathbf{t}_i \rangle \mathbf{n}_i) \tag{32}$$

Eqs. (30), (31) with (28) constitute a numerical model for effective elastic moduli. Eq. (32) with Eq. (28) can be used to calculate effective in-plane shear modulus for various elliptical RVEs. For each realization of the random distributions of N parallel microcracks in a circular or elliptic sub-region, effective elastic moduli are computed. Then, the desired effective moduli can be evaluated as the mean values of several realizations.

A standard random number generator which randomly and successively generates the center of each crack is used to generate 200 cracks with the same length in an elliptic region to form samples corresponding to each crack density. Ten samples of the crack distributions corresponding to each crack density and each elliptical region are generated. Following Kachanov (1992), the spacing between cracks is kept no smaller than 0.02 of the crack length. Fig. 2 shows one of the samples with aspect ratio being 0.5. For the solids with parallel tunnel cracks, the generalized non-interacting solutions for effective G_{12} are evaluated for various aspect ratio a of elliptic RVE and compared with the numerical calculations. For the elliptical RVE with the aspect ratios being 0.1, 0.5 and 1, the effective in-plane shear modulus G_{12} with $v_0 = 0.3$ are calculated and plotted in Fig. 3.

The effective moduli E_1 and G_{12} with the crack densities $\rho = 0.1$, 0.2, 0.3, 0.4, 0.5 and 0.6, respectively, are also evaluated and illustrated in Figs. 4 and 5 together with some existing solutions including the conventional non-interacting solution, self-consistent method (Hoenig, 1979; Hu and Huang, 1993) and differential method (Zimmerman, 1985; Hashin, 1988). The present numerical results lie between the conventional non-interacting solution and the differential method.

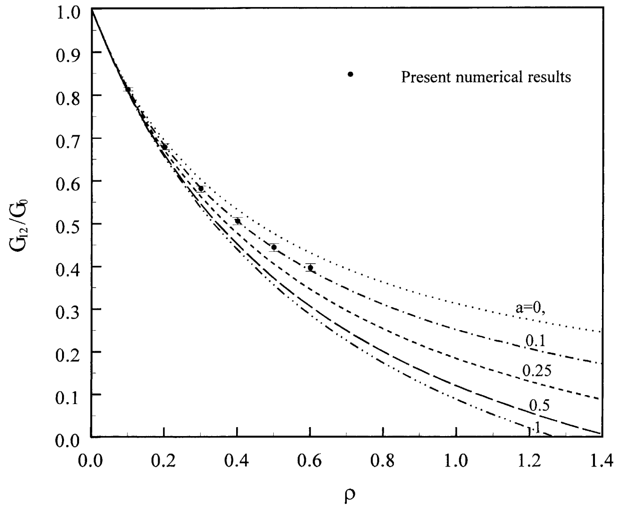

Fig. 3. Normalized effective in-plane shear modulus of solids with parallel tunnel cracks (a = aspect ratio of the elliptic RVE).

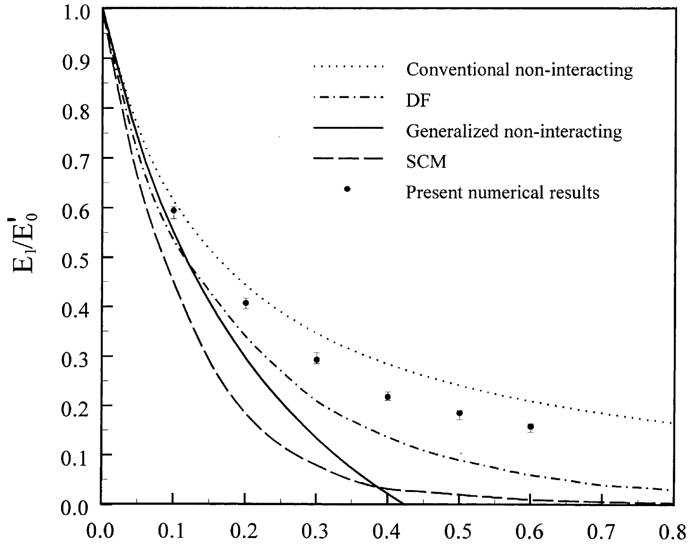

Fig. 4. Normalized effective plane strain modulus normal to the crack direction of solids with parallel tunnel cracks.

5. Two-phase composites with randomly dispersed spherical inhomogeneities

5.1. Generalized non-interacting solution

For isotropic effective moduli **C**, by taking far-field stress σ^0 as hydrostatic tension σ^0_K with $\sigma^0_{Kij} = \sigma^0\delta_{ij}$ and shear stresses σ^0_G with $\sigma^0_{G12} = \sigma^0_{G21} = \tau^0$ and other $\sigma^0_{Gij} = 0$, respectively, and considering the spherical RVE, two independent equations of Eq. (8b) for effective bulk and shear moduli can be obtained as

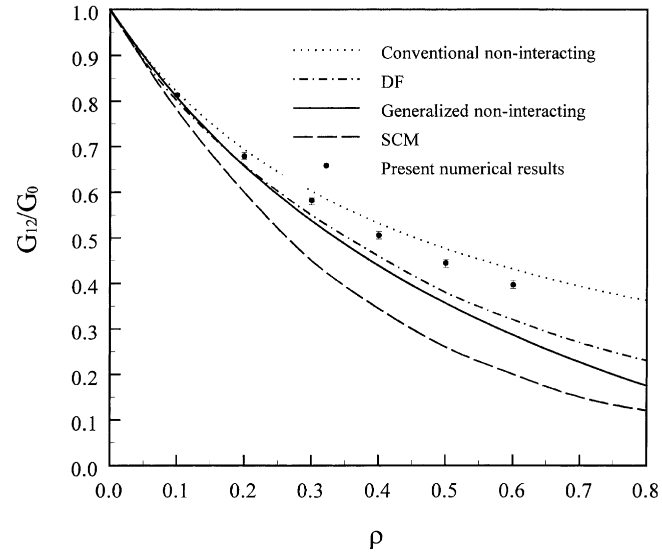

Fig. 5. Normalized effective in-plane shear modulus of solids with parallel tunnel cracks.

$$\frac{1}{K_0}\frac{K-K_0}{K_0+\xi(K-K_0)} = -\frac{1}{(\sigma^0)^2}\sigma_K^0 : \sum_{i=1}^{N}\phi_i\bar{\varepsilon}_i^* \tag{33}$$

$$\frac{1}{G_0}\frac{G-G_0}{G_0+\eta(G-G_0)} = -\frac{1}{(\tau^0)^2}\sigma_G^0 : \sum_{i=1}^{N}\phi_i\bar{\varepsilon}_i^* \tag{34}$$

where $\xi = (1+\nu_0)/[3(1-\nu_0)]$, $\eta = (8-10\nu_0)/[15(1-\nu_0)]$, K_0 and G_0 denote bulk and shear moduli of the matrix of the composite, and K and G denote those of the effective medium. Using the non-interacting approximation, Eqs. (33) and (34) become

$$\frac{K-K_0}{K_0+\xi(K-K_0)} = \phi_1\frac{K_1-K_0}{K_0+\xi(K_1-K_0)} \tag{35}$$

$$\frac{G-G_0}{G_0+\xi(G-G_0)} = \phi_1\frac{G_1-G_0}{G_0+\xi(G_1-G_0)} \tag{36}$$

where ϕ_1, K_1 and G_1 denote the volume fraction, bulk and shear moduli of the spherical inhomogeneities. Note that the solution expressed in Eqs. (35) and (36) can also be obtained from the generalized non-interacting solution, i.e., Eq. (11).

Subsequently, K and G can be extracted from Eqs. (35) and (36) as

$$K = K_0\left[1+\frac{\phi_1(K_1-K_0)}{K_0+\xi(1-\phi_1)(K_1-K_0)}\right] \tag{37}$$

$$G = G_0\left[1+\frac{\phi_1(G_1-G_0)}{G_0+\eta(1-\phi_1)(G_1-G_0)}\right] \tag{38}$$

If the matrix is the softer (or harder) phase, the above K and G are identical to the lower (or upper) bounds derived by Hashin and Shtrikman (1963). These solutions are also coincided with those evaluated by the

Mori–Tanaka solution (see Weng, 1984 and 1990), the double-inclusion model by Hori and Nemat-Nasser (1993), the "non-interacting" solution by Ju and Chen (1994a) and the estimate of Hashin–Shtrikman type by Ponte Casta ñ eda and Willis (1995).

It is interesting to see from Eqs. (35) and (36) that a symmetric structure is implied in these solutions.

5.2. Numerical calculation

The accuracy of effective elastic moduli depends on how precisely to solve the integral Eqs. (5) and (6) for $\bar{\varepsilon}_i^*$ under the corresponding far-field stresses. Rodin and Hwang (1991) converted approximately the integral equations (5) and (6) into the system of linear algebraic equations using Kachanov's idea (Kachanov, 1987). In this study, the system of linear algebraic equations is rederived using an alternative approach.

Substituting Eq. (6) into Eq. (5) leads to

$$\mathbf{C}_i\left[\varepsilon^0+\sum_{j\neq i}\int_{V_j}\mathbf{s}(\mathbf{x}-\mathbf{x}'):(\bar{\varepsilon}_j^*+\Delta\varepsilon_j^*(\mathbf{x}'))\,\mathrm{d}\mathbf{x}'+\int_{V_i}\mathbf{s}(\mathbf{x}-\mathbf{x}'):\varepsilon_i^*(\mathbf{x}')\,\mathrm{d}\mathbf{x}'\right]$$
$$=\mathbf{C}_0:\left[\varepsilon^0+\sum_{j\neq i}\int_{V_j}\mathbf{s}(\mathbf{x}-\mathbf{x}'):(\bar{\varepsilon}_j^*+\Delta\varepsilon_j^*(\mathbf{x}'))\,\mathrm{d}\mathbf{x}'+\int_{V_i}\mathbf{s}(\mathbf{x}-\mathbf{x}'):\varepsilon_i^*(\mathbf{x}')\,\mathrm{d}\mathbf{x}'-\varepsilon_i^*(\mathbf{x})\right]\quad\text{in } V_i \tag{39}$$

where $\Delta\varepsilon_j^*(\mathbf{x}')$ is the deviation of the eigenstrain $\varepsilon_j^*(\mathbf{x}')$ from its volume average value over V_i, $\bar{\varepsilon}_j^*$, that is,

$$\Delta\varepsilon_j^*(\mathbf{x}')\equiv\varepsilon_j^*(\mathbf{x}')-\bar{\varepsilon}_j^* \tag{40}$$

By taking the volume average of Eq. (39) over V_i, it becomes

$$\mathbf{C}_i\left(\varepsilon^0+\sum_{j\neq i}\overline{\mathbf{D}}^{ji}:\bar{\varepsilon}_j^*+\mathbf{S}_0:\bar{\varepsilon}_i^*\right)=\mathbf{C}_0\left(\varepsilon^0+\sum_{j\neq i}\overline{\mathbf{D}}^{ji}:\bar{\varepsilon}_j^*+\mathbf{S}_0:\bar{\varepsilon}_i^*-\bar{\varepsilon}_i^*\right)+\delta\bar{\varepsilon}_i^*\quad\text{in } V_i \tag{41}$$

with

$$\delta\bar{\varepsilon}_i^*=(\mathbf{C}_0-\mathbf{C}_i):\sum_{j\neq i}\frac{1}{V_i}\int_{V_i}\int_{V_j}\mathbf{s}(\mathbf{x}-\mathbf{x}'):(\varepsilon_j^*(\mathbf{x}')-\bar{\varepsilon}_j^*)\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} \tag{42}$$

where $\overline{\mathbf{D}}^{ji}$ is the volume average of $\mathbf{D}^{ji}$ over V_i, and expressed as (Rodin and Hwang, 1991),

$$\overline{\mathbf{D}}^{ji}=\overline{\mathbf{D}}^{ji}(\mathbf{x}_0-\mathbf{x}_0')+\frac{a_j^3a_i^2}{30(1-\nu)}\nabla\nabla\nabla\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\bigg|_{\mathbf{x}-\mathbf{x}'=\mathbf{x}_0-\mathbf{x}_0'} \tag{43}$$

where a_i, a_j and $\mathbf{x}_0$, $\mathbf{x}_0'$ are the radii and centers of the ith and jth inhomogeneities, respectively; and the following results from Eshelby (1957,1959) have been used (see Rodin and Hwang, 1991):

$$\int_{V_j}\mathbf{s}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}\mathbf{x}'=\begin{cases}\mathbf{S}_0 & \text{if}\quad j=i\\ \mathbf{D}^{ji}(\mathbf{x}-\mathbf{x}') & \text{if}\quad j\neq i\end{cases} \tag{44}$$

Tensor $\mathbf{S}_0$ is the well-known Eshelby's tensor associated with spherical inclusion. Tensor $\mathbf{D}^{ji}$ is expressed in terms of potentials ϕ and ψ as (Eshelby, 1959)

$$\mathbf{D}_{mnpq}^{ji}=\frac{1}{8\pi(1-\nu)}\left\{\psi_{,pqmn}-2\nu\delta_{pq}\phi_{,mn}-(1-\nu)\left[\delta_{mq}\phi_{,np}+\delta_{nq}\phi_{,mp}+\delta_{mp}\phi_{,nq}+\delta_{np}\phi_{,mq}\right]\right\} \tag{45}$$

with

$$\phi = \frac{4\pi a_j^3}{3}\frac{1}{|\mathbf{x}-\mathbf{x}'|} \quad \text{and} \quad \psi = \frac{4\pi a_j^3}{3}|\mathbf{x}-\mathbf{x}'| + \frac{4\pi a_j^5}{15}\frac{1}{|\mathbf{x}-\mathbf{x}'|} \tag{46}$$

By neglecting $\delta\bar{\varepsilon}_i^*$ in Eq. (41), a system of $6N$ linear algebraic equations for the eigenstrains $\bar{\varepsilon}_i^*$ are

$$\mathbf{C}_i\left(\varepsilon^0 + \sum_{j\neq i}\overline{\mathbf{D}}^{ji} : \bar{\varepsilon}_j^* + \mathbf{S}_0 : \bar{\varepsilon}_i^*\right) = \mathbf{C}_0\left(\varepsilon^0 + \sum_{j\neq i}\overline{\mathbf{D}}^{ji} : \bar{\varepsilon}_j^* + \mathbf{S}_0 : \bar{\varepsilon}_i^* - \bar{\varepsilon}_i^*\right) \quad \text{in } V_i \ (i = 1, 2, \ldots N) \tag{47}$$

It should be noted that Eq. (47) is consistent with that of Rodin and Hwang (1991). Since $\int_{V_j}(\varepsilon_j^*(\mathbf{x}') - \bar{\varepsilon}_j^*)\,d\mathbf{x}' \equiv 0$, $\delta\bar{\varepsilon}_i^*$ in Eq. (42) is assumed to be small and it is neglected in the present study. Besides, since the microscopic geometry of the spherical inhomogeneities can be unambiguously specified, the validity of the numerical method can be tested by comparing with the results which is obtained with more involved numerical techniques such as the finite element method. Rodin and Hwang (1991) compared the potential energy changes due to imbedding two equal voids into an infinite solid subjected to uniaxial and triaxial far-field tensions. They concluded that the results predicted by the numerical method and the FEM, respectively, are very agreeable each other even for the distance of the two voids are small to 0.1% of their radius. Therefore, the numerical methods may produce relatively accurate solutions for the unknown $\bar{\varepsilon}_i^*$ associated with the problem of an infinite matrix containing N spherical inhomogeneities.

Eqs. (33) and (34) in conjunction with Eq. (47) form a numerical model to evaluate the effective bulk and shear moduli of composites with randomly distributed spherical inhomogeneities. The resulting effective moduli can be taken as the mean value of several realizations of N inhomogeneities. If the number of the inhomogeneities is taken large enough, it can be anticipated that the scatters of the results predicted by different realizations will be small.

For the cases of spherical pores and rigid spheres, Eq. (47) becomes

$$\varepsilon^0 + \sum_{j\neq i}\overline{\mathbf{D}}^{ji} : \bar{\varepsilon}_j^* + \mathbf{S}_0 : \bar{\varepsilon}_i^* - \bar{\varepsilon}_i^* = \mathbf{0} \quad \text{in } V_i \ (i = 1, 2, \ldots N) \tag{48}$$

and

$$\varepsilon^0 + \sum_{j\neq i}\overline{\mathbf{D}}^{ji} : \bar{\varepsilon}_j^* + \mathbf{S}_0 : \bar{\varepsilon}_i^* = \mathbf{0} \quad \text{in } V_i \ (i = 1, 2, \ldots N) \tag{49}$$

The standard random number generator was used to generate a realization of random distributions of N spherical inhomogeneities. Each spherical inhomogeneity is randomly and successively put into the spherical sub-region of an infinite matrix corresponding to a volume fraction. However, as pointed out by Rodin and Hwang (1991), when the volume fraction is larger than 0.3, it is extremely difficult to generate a random distribution of spherical inhomogeneities with a common radius. Actually, when the volume fraction is larger than 0.3, the spheres formerly generated have inappropriately occupied some locations so that the latter spherical inhomogeneities with the common size can not squeeze into the spaces between the former spherical inhomogeneities. Following Rodin and Hwang (1991), 108 small spherical inhomogeneities with radius being 0.6 and 72 large spherical inhomogeneities with radius being 1 are randomly generated in a spherical region with a radius corresponding to the volume fraction 0.4. However, it is noted that Rodin and Hwang (1991) randomly generated these spheres in a cubic region. Besides, 160 spherical inhomogeneities with the common size are randomly generated in spherical regions corresponding to volume fraction 0.1, 0.2 and 0.3. For each volume fraction, 10 samples of the random distributions are produced and calculated.

For verification studies, effective moduli of a sintered glass containing spherical pores ($\nu_0 = 0.193$) and epoxy containing pores ($\nu_0 = 0.4$) are calculated by the present method and compared with the existing experimental data (Walsh et al., 1965; Ishai and Cohen, 1967) and analytic solutions (Weng, 1990;

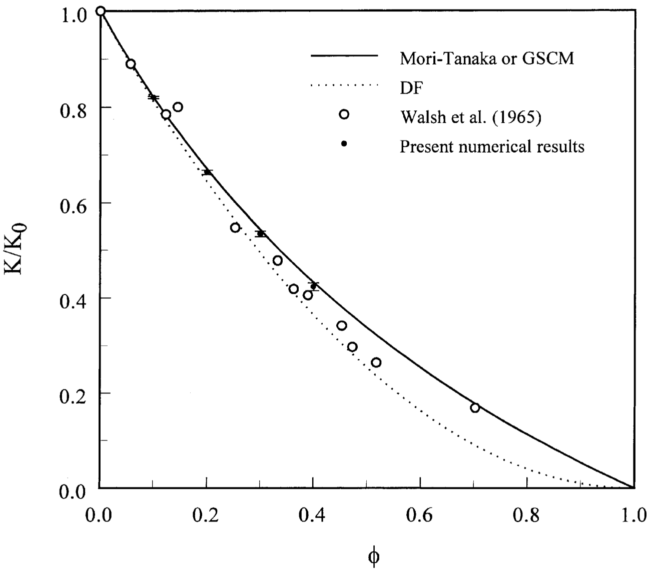

Fig. 6. Normalized Young's modulus of porous materials ($v_0 = 0.193$).

Zimmerman, 1991; Christensen, 1990). The bulk modulus of the sintered glass containing spherical pores was obtained experimentally by Walsh et al. (1965), while the Young's modulus of the epoxy containing pores was given by Ishai and Cohen (1967). As shown in Figs. 6 and 7, the numerical results are agreeable with the experimental data by Walsh et al. (1965) and Ishai and Cohen (1967).

For two-phase composites with spherical voids, the Mori–Tanaka solution (Weng, 1990) and the differential method (Zimmerman, 1991) are

$$K = K_0\left[1 - \frac{3(1-v_0)\phi_1}{3(1-v_0) + (1+v_0)(1-\phi_1)}\right] \tag{50}$$

$$G = G_0\left[1 - \frac{15(1-v_0)\phi_1}{15(1-v_0) + (8-10v_0)(1-\phi_1)}\right] \tag{51}$$

and

$$\frac{G}{K} = \frac{3}{4} + \frac{3(1-5v_0)}{4(1+v_0)}\left(\frac{G}{G_0}\right)^{3/5} \tag{52}$$

$$\frac{G}{G_0} = (1-\phi_1)^2\left(\frac{2(1+v_0) + (1-v_0)(G/G_0)^{3/5}}{3(1-v_0)}\right)^{1/3} \tag{53}$$

The effective shear modulus of composite composed of an incompressible matrix and rigid spheres is also calculated, which is mathematically analogous to the relative viscosity of Newtonian fluids containing rigid spheres. For two-phase composites composed of incompressible matrix and rigid spheres, the expressions of effective shear modulus predicted by the Mori–Tanaka solution (Weng, 1990) and the differential method (Zimmerman, 1991) reduce to

$$G/G_0 = (1 + 1.5\phi_1)/(1-\phi_1) \tag{54}$$

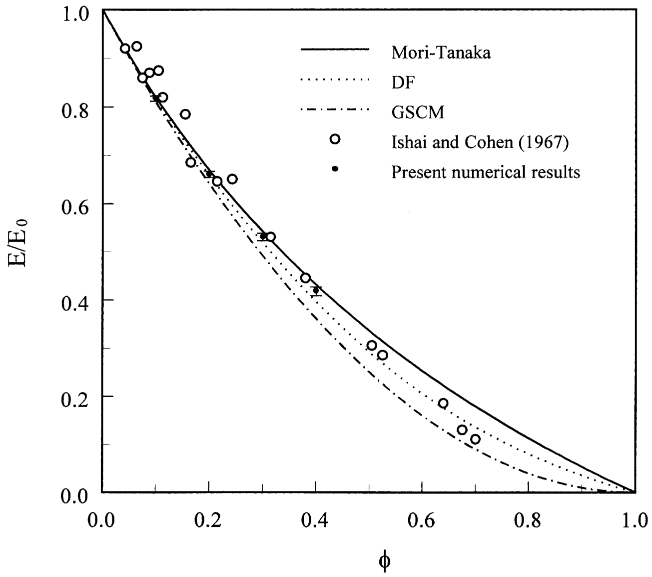

Fig. 7. Normalized Young's modulus of porous materials ($\nu_0 = 0.4$).

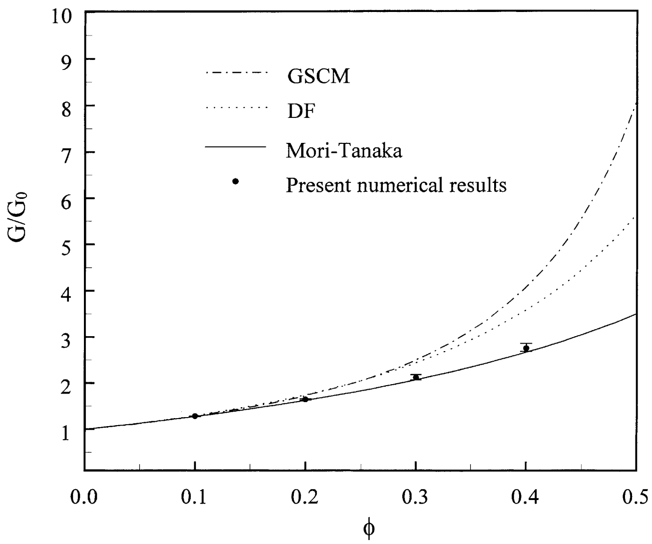

Fig. 8. Normalized effective shear modulus of composites composed of an incompressible matrix and rigid spheres.

and

$$G/G_0 = (1 - \phi_1)^{-2.5} \tag{55}$$

As shown in Fig. 8, the numerical results lie between the Mori–Tanaka solution and the different method but closer to the former.

6. Conclusion

An effective inclusion model is proposed based on the new energy balance equation. The generalized non-interacting solution for effective moduli of heterogeneous materials is obtained by solving the problem of an infinite matrix containing N inhomogeneities in an ellipsoidal sub-region. It is shown that the generalized non-interacting solution coincides with the estimates of the Hashin–Shtrikman type presented by Ponte Casta ñ eda and Willis (1995). The generalized non-interaction solutions completely neglect the interactions among cracks or spherical inhomogeneities while the numerical results account for the interactions. For the crack problem, it is shown that the numerical results for effective moduli of the plane strain problem of isotropic solids with parallel tunnel cracks is independent of the shape of RVEs and lie between the predictions by the conventional non-interaction solution and the differential method. For the two porous materials, the present numerical results are agreeable with the experimental data obtained by Walsh et al. (1965) and Ishai and Cohen (1967) for the volume fraction ratio up to 0.4 and lie between the Mori–Tanaka solution and the different method but closer to the former. The effective shear modulus of the composite with rigid spheres is also calculated by the present method. The results are closer to the generalized non-interacting solution or the Mori–Tanaka solution compared with the differential method.

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